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An internal solution in general relativity

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Abstract. A solution for the interior metric of a sphere has been obtained with the effective mass density as a variable quantity.

1. Introduction

Whittaker (1935) has pointed out that the effective mass density governing gravitational attraction is not ρ but $\rho+3p/c^2$ where ρ is the mass density and p is the pressure. Whittaker (1968) solved Einstein's field equations for the interior metric of a fluid sphere assuming $\rho+3p/c^2$ to be a constant. However a more general case will involve a form of this quantity varying with the radial coordinate. Here we have obtained a nonsingular solution of Einstein's field equations for the interior metric of a fluid sphere with a variable form of the effective mass density. We have assumed here that the metric coefficient $b = f(r) = \frac{1}{2}k_1r^2+k_2$ where $-b = g_{44}$ so that $\rho+3p/c^2$ varies with radial coordinate.

2. The field equations and their solutions

We assume a metric of the form

$$\mathrm{d}s^2 = a(r)\,\mathrm{d}r^2+r^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2)-b(r)c^2\,\mathrm{d}t^2. \tag{1}$$

The equations to be satisfied are then (Møller 1952, p 329)

$$\frac{\mathrm{d}p}{\mathrm{d}r}+(\rho c^2+p)\frac{b'}{2b} = 0 \tag{2}$$

$$\frac{b'}{abr}-\frac{1}{r^2}\left(1-\frac{1}{a}\right)+\Lambda = kp \tag{3}$$

$$\frac{a'}{a^2r}+\frac{1}{r^2}\left(1-\frac{1}{a}\right)-\Lambda = k\rho c^2 \tag{4}$$

where Λ is the cosmological constant and the prime denotes differentiation with respect to r. In the following solutions we have taken $\Lambda = 0$ and $k = 8\pi$. Now we assume

$$b = \tfrac{1}{2}k_1r^2+k_2 \tag{5}$$

where k_1 and k_2 are constants. Adding equations (3) and (4) we get

$$\rho c^2 + p = \frac{1}{8\pi ar}\left(\frac{a'}{a} + \frac{b'}{b}\right). \tag{6}$$

The equation (6) can be rewritten as

$$\rho c^2 + 3p = \frac{1}{8\pi ar}\left(\frac{a'}{a} + \frac{b'}{b} + \frac{1}{r} - \frac{b''}{b'}\right) + 2p \tag{7}$$

since

$$\frac{1}{r} - \frac{b''}{b'} = 0$$

from the assumption (5). Now, from equation (2) using equation (7) we get

$$\frac{\mathrm{d}p}{\mathrm{d}r} + \left[\frac{1}{8\pi ar}\left(\frac{a'}{a} + \frac{b'}{b} + \frac{1}{r} - \frac{b''}{b'}\right)\right]\frac{b'}{2b} = 0$$

which on integration leads to

$$p = \frac{1}{16\pi}\left(\frac{b'}{abr}\right) + k_3 \tag{8}$$

where k_3 is the constant of integration. From equation (3) using equations (5) and (8) we get

$$a = \frac{k_1 r^2 + k_2}{(1 + 8\pi k_3 r^2)(\frac{1}{2}k_1 r^2 + k_2)}. \tag{9}$$

Hence from equation (8) using the values of a and b from equations (9) and (5) respectively, we get

$$p = \frac{1}{16\pi}\,\frac{1 + 8\pi k_3 r^2}{r^2 + (k_2/k_1)} + k_3 \tag{10}$$

and from equation (4) using equation (9) we get

$$\rho c^2 = \frac{1}{8\pi}\left(\frac{-12\pi k_1^2 k_3 r^4 + (\frac{1}{2}k_1^2 - 28\pi k_1 k_2 k_3)r^2 + (\frac{3}{2}k_1 k_2 - 24\pi k_2^2 k_3)}{(k_1 r^2 + k_2)^2}\right). \tag{11}$$

From equations (10) and (11) we obtain $\rho c^2 + 3p$ as a variable quantity:

$$\rho c^2 + 3p = \frac{1}{16\pi}\left(\frac{48\pi k_1^2 k_3 r^4 + (4k_1^2 + 64\pi k_1 k_2 k_3)r^2 + 6k_1 k_2}{(k_1 r^2 + k_2)^2}\right). \tag{12}$$

Since at the boundary $p = 0$ we have from equation (10)

$$r_1 = \left(\frac{-(16\pi k_2 k_3 + k_1)}{24\pi k_1 k_3}\right)^{1/2} \tag{13}$$

where r_1 is the boundary. To make r_1 real, k_3 should be negative and $k_1 > 16\pi|k_3|k_2$. With the above conditions ρc^2, p and $\rho c^2 + 3p$ are all positive.

3.

The values at the centre of the sphere are

$$p_0 = \frac{1}{16\pi}\frac{k_1}{k_2}+k_3 \tag{14}$$

$$\rho_0 c^2 = \frac{3}{16\pi}\frac{k_1}{k_2}-3k_3 \tag{15}$$

$$\rho_0 c^2+3p_0 = \frac{3}{8\pi}\frac{k_1}{k_2}. \tag{16}$$

Hence from above it shows that $\rho_0 c^2, p_0$ and $\rho_0 c^2+3p_0$ are all positive. Hence from equations (14) and (15) since k_3 is negative

$$\rho_0 c^2 > 3p_0. \tag{17}$$

4.

Now, for the exterior solution, we know (Møller 1952, p 326)

$$b(r) = \frac{1}{a(r)} = 1-\frac{2m}{r}. \tag{18}$$

As $a(r)$ and $b(r)$ must be continuous and $ab = 1$ for the exterior solution, we have using equations (5) and (9),

$$1 = \frac{k_1 r_1^2+k_2}{1+8\pi k_3 r_1^2} \tag{19}$$

from which we get

$$k_3 = -\left(\frac{[(2k_1-k_1k_2)^2+8k_1^2k_2]^{1/2}-(2k_1-k_1k_2)}{32\pi k_2}\right). \tag{20}$$

This value of k_3 makes r_1 real and the condition $k_1 > 16\pi|k_3|k_2$ is satisfied, provided $1 > k_2 > 0$. Hence from equations (18) and (19),

$$\frac{2m}{r_1} = \tfrac{1}{2}k_1 r_1^2 - 8\pi k_3 r_1^2. \tag{21}$$

It shows that m is positive since k_3 is negative. It can also be shown that equation (21) can be expressed in terms of pressure and density as follows

$$\frac{2m}{r_1} = 8\pi r_1^2\{b_0 p_0 - [\tfrac{2}{3}p_0 - \tfrac{1}{6}(\rho_0 c^2+p_0)](b_0+1)\} \tag{22}$$

where b_0 is the value of b at $r = 0$.

References

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